Calculating, with income effects, the compensating variation for a state change

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Abstract

One can easily obtain exact closed-form solutions for the compensating variation (and equivalent variation) in the presence of income effects when the policy being evaluated can be described as a change in the state of the world and one is willing to assume the policy change does not change the individual's epsilon draw. Alternatively, if one assumes the policy changes the epsilon draw, the expectation of the compensating variation is a complicated integral, typically without a closed-form. The assumption that the policy does not affect one's epsilon draw is common, and often reasonable, but little discussed.

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Consider the very common practice of estimating an individual's compensating variation, cv, or $E[cv]$, for some policy using the estimated parameters of an indirect utility function(s), each with an additive random component. Separate in your mind the issue of estimating the parameters in the utility function(s) from the issue of estimating the cv for some policy after the parameters are estimated. This note is concerned only with the latter problem.

More specifically, this note is concerned solely with calculating an individual's cv for a change in the *state* of the world, a *state change*. A *state* is characterized by the levels of all of the exogenous variables that determine utility: an indirect utility function is assumed that identifies maximum utility as a function of the state.[1] A change from the world as it exists to a world with

[1] It is critical to distinguish between the indirect utility function and a conditional indirect

less to spend on goods and less air pollution is the sort of change that environmental economists often want to value, and the type of change valued here. This type of valuation exercise is common in environmental economics, health is another obvious example: one's *cv* for an improved health state. Referendum CVM questions are designed to value state changes (the world before and after the policy).

In explanation, assume one has estimated the parameters in the indirect utility function

$$u = f(y - p) + \beta q + \varepsilon \tag{1}$$

where q represents the quality of the exogenous state, ε is a random draw from some distribution, ε is known to the individual but not to the researcher, y is the individual's income, and p is the cost of maintaining this state, so $y - p$ is the expenditure on the numeraire. The variable q can be either a scalar or vector. The individual i subscript on u, y, p, q and ε are suppressed. There is no need for an alternative-specific subscript - Equation 1 reflects that one has optimized given the price and attributes of the state - it is an indirect utility function, not a conditional indirect utility function.

The form of $f(y-p)$ determines how income affects the *cv*: income effects are present when utility is a nonlinear function of expenditures on the numeraire. Let (p^0, q^0) denote the initial state and (p^1, q^1) the proposed state. For example, u^0 might represent maximum utility before air quality has been improved and u^1 maximum utility after air quality and taxes have both increased.

Assume, for calculation of the *cv*, that the epsilon draw is *state independent* ($\varepsilon^0 = \varepsilon^1 = \varepsilon$): the unobserved component of the individual's utility is the same in both states - the idiosyncratic component of one's preferences don't change when a policy changes the state. The implications of dropping this assumption are discussed below.

In Equation 1, ε is a scalar, not a vector. We find the assumption of state independence generally plausible. As explained below, the assumption that for welfare calculations the epsilon draw is state independent is a standard assumption, but also one that is rarely discussed.

Summarizing our findings, when one is valuing a state change and one makes the state-independence assumption, valuation simplifies in two significant ways: one can directly estimate the *cv* rather than being limited to estimating the $E[cv]$, and simple closed-form solutions exist for the *cv* (and *ev*) for many common specifications of income effects.

Alternatively, if one drops the state-independence assumption, *cv* varies widely across individuals as a function of their $(\varepsilon^1 - \varepsilon^0)$ draws, and some individuals' *cv* will be of the "wrong" sign (negative for a quality increase and price decrease). If there are no income effects, the formula for the $E[cv]$ is the formula for the *cv* when $\varepsilon^1 = \varepsilon^0$ is assumed. But, if there are income effects, the density

utility function. The latter identifies maximimum utility when the individual is constrained to consume a specific alternative (e.g. choose alternative j from among J possible choices). The former is the maximum of the J conditional indirect utility functions.

function of the cv is nonlinear in $(\varepsilon^1 - \varepsilon^0)$ and $E[cv]$ is a complicated integral, so dropping the state-independence assumption has significant consequences when there are income effects.

An aside - state independence in the J alternative framework: Contrast the indirect utility function for a state, Equation 1, with a situation where one models J alternatives in each state, one alternative must be chosen, and one specifies and estimates a conditional indirect utility function for each alternative, $u_j = v_j + \varepsilon_j$ where the subscript refers to the alternative, not the individual, and one assumes $\Pr(\varepsilon_m \neq \varepsilon_n) = 1$: no two alternatives have the exact same epsilon. For the calculation of $E[cv]$ for a change in one or more of the J alternatives, the state independence assumption is $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$ where $\boldsymbol{\varepsilon}$ is a vector with J elements.

The assumption that $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$ is plausible and is likely implicitly held by many of us: without this assumption the cv associated with an improvement in one or more of the J alternatives could be negative if one's quirkiness (epsilon vector) is policy specific - one might choose alternative m, but after its quality improves, not choose it.

Dagsvik and Karlström (2005), who are the first to derive a formula for $E[cv]$ for J-alternative discrete-choice models with income effects, explicitly assume $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$ and discuss it (few practitioners note or discuss the assumption), "it is assumed that the random terms $\{\varepsilon_k\}$ are not affected by the policy intervention. This seems reasonable if the error terms $\{\varepsilon_k\}$ characterize tastes. It is less reasonable if $\{\varepsilon_k\}$ also include unmeasured attributes of alternatives, which may be altered by policy."

The cv for the change in the characteristics or costs of one or more of the J alternatives is a function of the individual's $\boldsymbol{\varepsilon}$ draw, so the researcher can obtain only the $E[cv]$, not the cv itself. In explanation, the individual will often switch alternatives because of the policy, and since the epsilons differ across the alternatives (even assuming the epsilons are state independent), the epsilons do not cancel and the individual's cv is a function of his epsilon draw, which is not observed.

In the J case, if one assumes no income effects and, for example, a logit model, the $E[cv]$ is calculated with the well-known log-sum formula. If there are income effects, things are more difficult. One either has to approximate the $E[cv]$ using, for example, the representative consumer approximation (see Morey, Rowe and Watson 1993, and McFadden 1999), simulate the $E[cv]$ (see McFadden 1999 and Herriges and Kling 1999), or use the expected expenditure formula (Dagsvik and Karlström 2005).[2]

Note the possible schizophrenia between estimation of preference parameters and the calculation of the $E[cv]$. In the J alternative case with income effects, when one calculates the $E[cv]$ exactly one assumes $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$, but, prior to that, when one estimates the parameters in the J conditional indirect utility functions one assumes that each $\boldsymbol{\varepsilon}$ implicit in the data-generating-process is an

[2]The representative consumer approximation can be used whether one does, or does not, assume the epsilons are state independent. Programs used to simulate $E[cv]$ in the presence of income effects typically assume $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$. The derivation of the $E[cv]$ using the expected expenditure formula formally assumes $\boldsymbol{\varepsilon}^0 = \boldsymbol{\varepsilon}^1$ (Dagsvik and Karlström 2005).

independent draw, so not equal. The same disconnect can hold if one is calculating the $E[cv]$ for a state change assuming $\varepsilon^0 = \varepsilon^1$; estimation with data from choice pairs over states of the world requires that one assume each alternative in each pair is associated with a different ε draw. *End of aside.*

One last thing - approximations: When $E[cv]$ with income effects are discussed, there is usually a distinction made between an exact, closed-form for the $E[cv]$, typically difficult to calculate, and approximations to it. These discussions take place in the context of the J-alternative model.

This paper is not about approximations or the J-alternative model. For a state-change, if one adopts the state-independence assumption ($\varepsilon^0 = \varepsilon^1 = \varepsilon$), there is no reason to approximate the cv. If one does not make the state-independence assumption, calculation of the $E[cv]$ for a state change does not have a simple closed-form solution, and one might want to consider approximations. While this paper is not about approximations, one might ask whether the formulae presented next, which are exact if one assume state independence, are good approximations if one relaxes the assumption. Our sense is that they often will not be good approximations; this is briefly discussed.

1 Exact cv formulae for a state change

Returning to Equation 1, the compensating variation is that amount of money that when subtracted from the individual's income in the new state makes utility in the new state, with the subtraction, equal to utility in the original state. That is, the cv is that c such that

$$u^0 \equiv f(y - p^0) + \beta q^0 + \varepsilon = f(y - p^1 - c) + \beta q^1 + \varepsilon \tag{2}$$

By assumption, the two epsilons cancel, so the cv is not a function of the individual's epsilon draw; that is, one solves Equation 2 for the cv not the $E[cv]$. This is the case whether or not there are income effects. Note that the functional form of the cv is independent of how the ε are distributed. Rearranging Equation 2

$$\beta(q^1 - q^0) = f(y - p^0) - f(y - p^1 - c) \tag{3}$$

Equation 3 can be easily solved for many forms of $f(w)$.

Start with the case of no income effects $f(w) = \alpha w$, where α is the constant marginal utility of money. In which case,

$$cv = \frac{\beta(q^1 - q^0)}{\alpha} + (p^0 - p^1) \tag{4}$$

The cv is just the price change plus the quality change converted into money by dividing it by the constant marginal utility of money, not surprising.

Now consider some income-effects specifications ($f(w)$ a nonlinear function of w). If $f(w) = \alpha \ln w$, a one-parameter specification,

$$cv = (y - p^1) - e^{-(\frac{\beta(q^1 - q^0)}{\alpha})}(y - p^0) \tag{5}$$

Alternatively, if $f(w) = \alpha_0 w + \alpha_1 w^{.5}$

$$\begin{aligned} cv &= p^0 - p^1 + \frac{\beta(q^1 - q^0)}{\alpha_0} - \gamma\sqrt{(y - p^0)} - \frac{1}{2}\gamma^2 \\ &\quad + \gamma\sqrt{\left(\frac{\gamma^2}{2} - \frac{2\beta(q^1 - q^0)}{\alpha_0} - 2p^0 + 2\gamma\sqrt{(y - p^0)} + 2y\right)} \end{aligned} \tag{6}$$

where $\gamma = \alpha_1/\alpha_0$. If $\alpha_1 = 0$ (no income effects), Equation 6 simplifies, as it must, to Equation 4. If $\alpha_0 = 0$ and $\alpha_1 \neq 0$, $cv = p_0 - p_1 - (\frac{\beta(q^1-q^0)}{\alpha_1})^2 + \frac{2\beta(q^1-q^0)}{\alpha_1}\sqrt{(y - p_0)} = y - p_1 - \left(-\frac{\beta(q^1-q^0)}{\alpha_1} + \sqrt{y - p_0}\right)^2$.

Alternatively, consider $f(w) = \alpha_0 w + \alpha_1 w^2$, If $\gamma \neq 0$

$$cv = \frac{1}{2\gamma}\left(1 + 2\gamma y - 2\gamma p^1 - v\right) \tag{7}$$

where $v = \sqrt{\left(1 + 4\gamma y + 4\gamma^2 y^2 - 4\gamma\frac{\beta(q^1-q^0)}{\alpha_0} - 4\gamma p^0 - 8\gamma^2 y p^0 + 4\gamma^2 (p^0)^2\right)}$. And if $\gamma = 0$, $cv = \frac{\beta(q^1-q^0)}{\alpha_0} + p_0 - p_1$, as it must, the no income-effects specification. If $\alpha_0 = 0$ and $\alpha_1 \neq 0$, $cv = y - p^1 - \sqrt{\left(y^2 - 2yp^0 + (p^0)^2 - \frac{\beta(q^1-q^0)}{\alpha_1}\right)}$

For the final example, assume that the utility is a linear spline function of expenditures on the numeraire. That is, the marginal utility of expenditures on the numeraire. depends only on whether one is poor (ρ), rich (r) or middle class (m). In this case,

$$f(w) = \left\{ \begin{array}{ll} \alpha_0 w & \text{if } w \leq l_0 \\ \alpha_0 l_0 + \alpha_1(w - l_0) & \text{if } l_0 < w \leq l_1 \\ \alpha_0 l_0 + \alpha_1(l_1 - l_0) + \alpha_2(w - l_1) & \text{if } w > l_1 \end{array} \right\} \tag{8}$$

Assume the marginal utility of money declines in three steps as income increases ($\alpha_0 > \alpha_1 > \alpha_2 > 0$) and consider only a quality improvement ($q^1 > q^0, p^1 = p^0$).

If an individual is poor in the initial state, after paying compensation he is still poor. If initially middle class, after paying compensation he will either remain middle class or become poor. If initially rich, paying the compensation will cause him to remain rich, become middle class or become poor. Thus, there are six possible compensating variations: $cv_{\rho\rho}$, cv_{mm}, cv_{rr}, $cv_{m\rho}$, cv_{rm}, and $cv_{r\rho}$.

Substitute Equation 8 into Equation 3 and solve for c. The cv for the different income-category shifts are

$$cv_{\rho\rho} = \frac{\beta(q^1 - q^0)}{\alpha_0} \tag{9}$$

$$cv_{mm} = \frac{\beta(q^1 - q^0)}{\alpha_1} \tag{10}$$

$$cv_{rr} = \frac{\beta(q^1 - q^0)}{\alpha_2} \tag{11}$$

If paying the compensation causes the individual to move from category m to ρ, the cv is[3]

$$cv_{m\rho} = \frac{\beta(q^1 - q^0)}{\alpha_0} + \left(1 - \frac{\alpha_1}{\alpha_0}\right)(y - l_0 - p), \tag{12}$$

For the other shifts, the cv are

$$cv_{rm} = \frac{\beta(q^1 - q^0)}{\alpha_1} + \left(1 - \frac{\alpha_2}{\alpha_1}\right)(y - l_1 - p) \text{ , and} \tag{13}$$

$$cv_{r\rho} = \frac{\beta(q^1 - q^0)}{\alpha_0} + (y - l_0 - p) - \frac{\alpha_1}{\alpha_0}(l_1 - l_0) - \frac{\alpha_2}{\alpha_0}(y - l_1 - p) \tag{14}$$

The following algorithm applies: $cv = cv_{\rho\rho}$ if $(y - p) \leq l_0$. If initially $l_0 < (y - p) \leq l_1$, first calculate cv_{mm}, then if $(y - p - cv_{mm}) > l_0$, $cv = cv_{mm}$, if not $cv = cv_{mp}$. If initially, $(y-p) > l_1$, first calculate cv_{rr}. If $(y-p-cv_{rr}) > l_1$, $cv = cv_{rr}$; if not, calculate cv_{rm}, then if $(y - p - cv_{rm}) > l_0$, $cv = cv_{rm}$, if not $cv = cv_{r\rho}$.

Note that for some forms of $f(w)$, one will not be able to solve Equation 3 and derive a formula for the cv. But, in such cases, given the parameter estimates, the change in p and q, and each individuals income, one can numerically solve Equation 3 to obtain each individual's cv. Adamowicz et al. (1999) do this, but only for the average individual in the sample.

2 ev formulae for a state change

The ev for a change in states is defined as that e for which

$$u^1 \equiv f(y - p^1) + \beta q^1 + \varepsilon = f(y - p^0 + e) + \beta q^0 + \varepsilon \tag{15}$$

[3] Before the change, utility is $\beta q^0 + \alpha_0 l_0 + \alpha_1(y - l_0 - p)$, and after the change and compensation is paid, utility is $\beta q^1 + \alpha_0(y - p - c)$. Solve for c to get $CV_{m\rho}$.

Closed-formed solution exist for the ev for many specifications of $f(y-p)$, including all those presented above. For example, for the no income effects specification, $\alpha(y-p)$, as required

$$ev = cv = \frac{\beta(q^1 - q^0)}{\alpha} + (p^0 - p^1)$$

Alternatively, if $f(w) = \alpha \ln w$,

$$ev = e^{\frac{\beta(q^1 - q^0)}{\alpha}}(y - p^1) - (y - p^0) \tag{16}$$

which does not equal $cv = (y - p^1) - e^{-(\frac{\beta(q^1 - q^0)}{\alpha})}(y - p^0)$, Equation 5, unless there is only a change in prices ($p^1 \neq p^0$ but $q^1 = q^0$).

3 The cv for a state change can be used to bound the $E[cv]$ for many J-alternative policy scenarios

Even in those situations where one has specifically modeled the choice of each alternative in each state (a conditional indirect utility function for each alternative), the cv formula for a state change can be useful. Consider a situation where the individual faces some set of J alternatives and chooses one of them (e.g. which recreational site to visit). The researcher has estimated the parameters in the J conditional indirect utility functions with complicated income effects, and the researcher wishes to calculate the $E[cv]$ for a change in the costs and characteristics of one of those alternatives, but finds applying the simulation method or expected-expenditure method daunting.

One can utilize the state-to-state cv formula outlined above to obtain valuable bounding information on the $E[cv]$. Consider the following common example from NRDA. Pollutants have contaminated a site and the researcher's objective is to estimate use damages. The researcher can use the state-to-state cv formula to calculate each individual's cv associated with eliminating the injury, **conditional** on the individual visiting the site. This is the cv per user-day at that site. In explanation, the calculation artificially constrains the individual to visit the site both before and after the change, so the epsilon cancels out of Equation 2. For an improvement, the cv per user-day multiplied by the current number of user days, is a lower bound on the individual's $E[cv]$ for the elimination of the injuries. That is, one can always use the above state-to-state formulae to calculate the cv conditional on the choice of a specific alternative, one simply solves Equation 3 using the conditional indirect utility function for the chosen alternative.

4 Summarizing to here

Researchers often want to estimate the *cv* associated with a change in the state of the world. This can be accomplished by specifying and estimating the parameters in the indirect utility function for a state, and then solving Equation 3 for c to obtain the *cv* formula for any change in states. With no income effects, the *cv* formula is simply the price change, plus the quality change converted into money by dividing it by the constant marginal utility of money - something well known, at least implicitly. What is not widely realized, is that when the utility is assumed some nonlinear function of expenditures on the numeraire. (income effects), simple closed-form formulae also exist, also for the *ev*, both as long as one is willing to assume the ε is not policy specific. A number of the formulae are reported above. So, one can, without hassle, estimate the exact *cv* and *ev* for a change in the state of the world, rather than approximate it, even when there are income effects.

While our results will be obvious to some, we have yet to find studies that assume income effects and derive and utilize exact formulae for the *cv* and *ev* of the sort derived above.

5 What if one doesn't assume $\varepsilon^0 = \varepsilon^1$?

Equation 3 is replaced by

$$\beta(q^1 - q^0) = f(y - p^0) - f(y - p^1 - c) + (\varepsilon^0 - \varepsilon^1) \tag{17}$$

with interesting implications. An individual's *cv* and $E[cv]$ now differ, and they can be of opposite signs. $E[cv]$ can be negative for price decreases or quality increases.

In the no-income effects case ($f(w) = \alpha w$)

$$cv = \frac{\beta(q^1 - q^0)}{\alpha} + (p^0 - p^1) + \frac{1}{\alpha}(\varepsilon^1 - \varepsilon^0) \tag{18}$$

and *cv* can be negative for a quality increase, or price decrease. The policy now changes the *cv* for two reasons: the change in p and q, and a random change that varies across individuals. Note that when there are no income effects, the distribution in the the *cv*'s is generated by a linear term, $\frac{1}{\alpha}(\varepsilon^1 - \varepsilon^0)$, and

$$E[cv] = \frac{\beta(q^1 - q^0)}{\alpha} + (p^0 - p^1) + \frac{1}{\alpha}(E[\varepsilon^1] - E[\varepsilon^0]) \tag{19}$$

which equals

$$E[cv] = \frac{\beta(q^1 - q^0)}{\alpha} + (p^0 - p^1) \tag{20}$$

as long as $E[\varepsilon^1] = E[\varepsilon^0]$. Note that the RHS of Equation 20 is the formula for the cv if one assumes $\varepsilon^1 = \varepsilon^0$, Equation 4

Things are more complicated when there are income effects. For example, if $f(w) = \alpha \ln w$

$$cv = (y - p^1) - (y - p^0)e^{-[\frac{\beta(q^1 - q^0)}{\alpha} + \frac{(\varepsilon^1 - \varepsilon^0)}{\alpha}]} \tag{21}$$

And

$$\begin{aligned} E[cv] &= \int [(y - p^1) - (y - p^0)e^{-[\frac{\beta(q^1 - q^0)}{\alpha} + \frac{v}{\alpha}]}]g(v)dv \\ &= (y - p^1) - (y - p^0)e^{-(\frac{\beta(q^1 - q^0)}{\alpha})} E[e^{-\frac{v}{\alpha}}] \end{aligned} \tag{22}$$

where $g(v)$ is the density function of $(\varepsilon^1 - \varepsilon^0)$.[4] Note that

$$E[cv] \neq (y - p^1) - (y - p^0)e^{-(\frac{\beta(q^1 - q^0)}{\alpha})} \tag{23}$$

where the RHS of Equation 23 is the cv formula if one assumes $\varepsilon^1 = \varepsilon^0$, Equation 5. Summarizing, when there are income effects the distribution of the cv is highly nonlinear in $(\varepsilon^1 - \varepsilon^0)$, and $E[cv]$ will often not have a closed-form solution.

From looking at some examples, it is our sense that formulae for the cv for a state change assuming $\varepsilon^0 = \varepsilon^1$ will often not be good approximations to the $E[cv]$ if state independence is dropped.

References

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[4] If, for example, one assumes each ε is an independent draw from an extreme-value distribution, cv will have a logistic distribution.